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ECS332 2012/1 Part I Dr.Prapun

1 Introduction to communication systems

1.1. Shannon's insight [8]:

> The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.

Definition 1.2. Figure 1 [8] shows a commonly used model for a (single-link or point-to-point) communication system. All information transmission systems involve three major subsystems–a transmitter, the channel, and a receiver.

(a) **Information** source: produce a **message**

- Messages may be categorized as **analog** (continuous) or **digital** (discrete).

(b) **Transmitter**: operate on the message to create a **signal** which can be sent through a channel

(c) **Channel**: the medium over which the signal, carrying the information that composes the message, is sent

- All channels have one thing in common: the signal undergoes **degradation** from transmitter to receiver.
    - Although this degradation may occur at any point of the communication system block diagram, it is customarily associated with the channel alone.

- This degradation often results from noise and other undesired signals or interference but also may include other distortion effects as well, such as fading signal levels, multiple transmission paths, and filtering.

(d) **Receiver**: transform the signal back into the message intended for delivery

(e) **Destination**: a person or a machine, for whom or which the message is intended

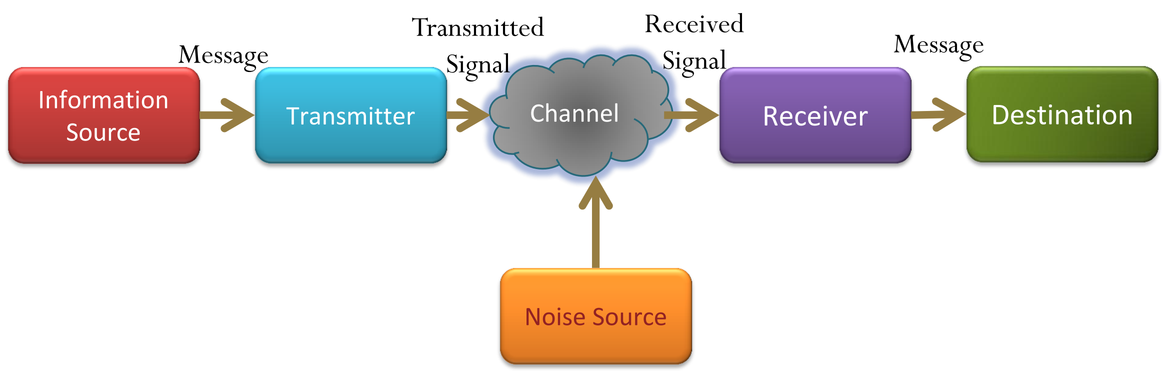

Figure 1: Schematic diagram of a general communication system

2 Frequency-Domain Analysis

Electrical engineers live in the two worlds, so to speak, of time and frequency. Frequency-domain analysis is an extremely valuable tool to the communications engineer, more so perhaps than to other systems analysts. Since the communications engineer is concerned primarily with signal bandwidths and signal locations in the frequency domain, rather than with transient analysis, the essentially steady-state approach of the (complex exponential) **Fourier series** and **transforms** is used rather than the Laplace transform.

2.1 Math background

2.1. ***Euler's formula***: $\boxed{e^{jx} = \cos x + j\sin x.}$

$$\cos(A) = \operatorname{Re}\left\{e^{jA}\right\} = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$
$$\sin(A) = \operatorname{Im}\left\{e^{jA}\right\} = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right).$$

2.2. We can use $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$ and $\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$ to derive many trigonometric identities.

Example 2.3. $\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$

2.4. Similar technique gives

(a) $\cos(-x) = \cos(x)$,

(b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$,

(c) $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, and

(d) the **product-to-sum formula**

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right). \tag{1}$$

2.2 Continuous-Time Fourier Transform

Definition 2.5. The (direct) **Fourier transform** of a signal $g(t)$ is defined by

$$G(f) = \int_{-\infty}^{+\infty} g(t) e^{-j2\pi ft} dt \tag{2}$$

This provides the frequency-domain description of $g(t)$. Conversion back to the time domain is achieved via the **inverse (Fourier) transform**:

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft} df \tag{3}$$

- We may combine (2) and (3) into one compact formula:

$$\boxed{\int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft} df = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi ft} dt.} \tag{4}$$

- We may simply write $G = \mathcal{F}\{g\}$ and $g = \mathcal{F}^{-1}\{G\}$.
- Note that $G(0) = \int g(t)dt$ and $g(0) = \int G(f)df$.

2.6. In some references[1], the (direct) Fourier transform of a signal $g(t)$ is defined by

$$G_2(\omega) = \int_{-\infty}^{+\infty} g(t) e^{-j\omega t} dt \tag{5}$$

[1] MATLAB uses this definition.

In which case, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega)\, e^{j\omega t} d\omega = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_2(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t} dt \tag{6}$$

- In MATLAB, these calculations are carried out via the commands `fourier` and `ifourier`.
- Note that $\hat{G}(0) = \int g(t)dt$ and $g(0) = \frac{1}{2\pi}\int G(\omega)d\omega$.
- The relationship between $G(f)$ in (2) and $G_2(\omega)$ in (5) is given by

$$G(f) = G_2(\omega)\big|_{\omega=2\pi f} \tag{7}$$

$$G_2(\omega) = G(f)\big|_{f=\frac{\omega}{2\pi}} \tag{8}$$

2.7. Q: The relationship between $G(f)$ in (2) and $G_2(\omega)$ in (5) is given by (7) and (8) which do not involve a factor of 2π in the front. Why then does the factor of $\frac{1}{2\pi}$ shows up in (6)?

Example 2.8. Rectangular and Sinc:

$$1\left[|t| \le a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{\sin(2\pi f a)}{\pi f} = \frac{2\sin(a\omega)}{\omega} = 2a\ \operatorname{sinc}(a\omega) \tag{9}$$

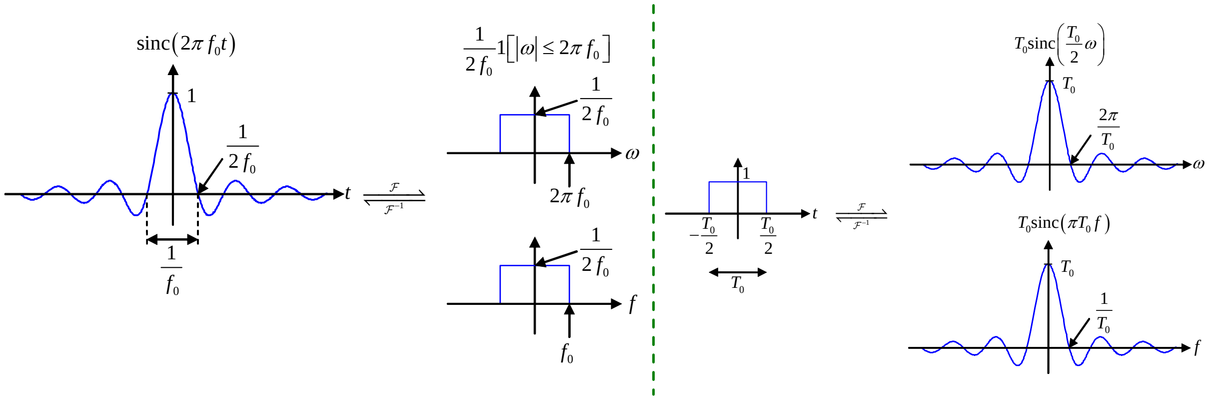

Figure 2: Fourier transform of sinc and rectangular functions

- By setting $a = T_0/2$, we have

$$1\left[|t| \leq \frac{T_0}{2}\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} T_0 \operatorname{sinc}(\pi T_0 f). \tag{10}$$

- In [4, p 78], the function $1[|t| \leq 0.5]$ is defined as the **unit gate** function $\operatorname{rect}(x)$.

Definition 2.9. The function $\operatorname{sinc}(x) \equiv (\sin x)/x$ is plotted in Figure 3.

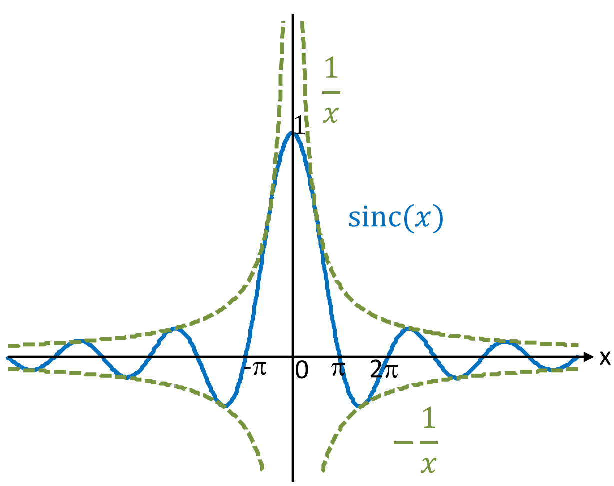

Figure 3: Sinc function

- This function plays an important role in signal processing. It is also known as the filtering or interpolating function.

- Using L'Hôpital's rule, we find $\lim_{x \to 0} \operatorname{sinc}(x) = 1$.
- $\operatorname{sinc}(x)$ is the product of an oscillating signal $\sin(x)$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, $\operatorname{sinc}(x)$ exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.
- In `MATLAB` and in [11, eq. 2.64], $\operatorname{sinc}(x)$ is defined as $(\sin(\pi x))/\pi x$. In which case, it is an even damped oscillatory function with zero crossings at integer values of its argument.

Definition 2.10. The (Dirac) **delta function** or (unit) impulse function is denoted by $\delta(t)$. It is usually depicted as a vertical arrow at the origin. Note that $\delta(t)$ is not a true function; it is undefined at $t = 0$. We define $\delta(t)$ as a generalized function which satisfies the **sampling property** (or **sifting property**)

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \tag{11}$$

for any function $\phi(t)$ which is continuous at $t = 0$. From this definition, It follows that

$$(\delta * \phi)(t) = (\phi * \delta)(t) = \int_{-\infty}^{\infty} \phi(\tau)\delta(t-\tau)d\tau = \phi(t) \tag{12}$$

where we assume that ϕ is continuous at t.

- Intuitively we may visualize $\delta(t)$ as an infinitely tall, infinitely narrow rectangular pulse of unit area: $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} 1\left[|t| \le \frac{\varepsilon}{2}\right]$.

2.11. Properties of $\delta(t)$:

- $\delta(t) = 0$ when $t \neq 0$.
  $\delta(t - T) = 0$ for $t \neq T$.
- $\int_A \delta(t)dt = 1_A(0)$.
  (a) $\int \delta(t)dt = 1$.
  (b) $\int_{\{0\}} \delta(t)dt = 1$.
  (c) $\int_{-\infty}^{x} \delta(t)dt = 1_{[0,\infty)}(x)$. Hence, we may think of $\delta(t)$ as the "derivative" of the unit step function $U(t) = 1_{[0,\infty)}(x)$.

- $\int \phi(t)\delta(t)dt = \phi(0)$ for ϕ continuous at 0.
- $\int \phi(t)\delta(t-T)dt = \phi(T)$ for ϕ continuous at T. In fact, for any $\varepsilon > 0$,

$$\int_{T-\varepsilon}^{T+\varepsilon} \phi(t)\delta(t-T)dt = \phi(T).$$

- $\delta(at) = \frac{1}{|a|}\delta(t)$. In particular,

$$\delta(\omega) = \frac{1}{2\pi}\delta(f) \tag{13}$$

and

$$\delta(\omega - \omega_0) = \delta(2\pi f - 2\pi f_0) = \frac{1}{2\pi}\delta(f - f_0), \tag{14}$$

where $\omega = 2\pi f$ and $\omega_0 = 2\pi f_0$.

Example 2.12. $\delta(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1$.

Example 2.13. $e^{j2\pi f_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \delta\left(f - f_0\right)$.

Example 2.14. $e^{j\omega_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi\delta\left(\omega - \omega_0\right)$.

Example 2.15. $\cos(2\pi f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(\delta\left(f - f_0\right) + \delta\left(f + f_0\right)\right)$.

2.16. Conjugate symmetry[2]: If $x(t)$ is **real**-valued, then $X(-f) = (X(f))^*$

Observe that if we know $X(f)$ for all f positive, we also know $X(f)$ for all f negative. Interpretation: Only half of the spectrum contains all of the information. Positive-frequency part of the spectrum contains all the necessary information. The negative-frequency half of the spectrum can be determined by simply complex conjugating the positive-frequency half of the spectrum.

2.17. ***Shifting*** properties

- ***Time-shift***:

$$g(t - t_1) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{-j2\pi f t_1} G(f)$$

  - Note that $|e^{-j2\pi f t_1}| = 1$. So, the spectrum of $g(t - t_1)$ looks exactly the same as the spectrum of $g(t)$ (unless you also look at their phases).

- ***Frequency-shift*** (or modulation):

$$e^{j2\pi f_1 t} g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G(f - f_1)$$

[2] Hermitian symmetry in [7, p 17].

2.18. Let $g(t)$, $g_1(t)$, and $g_2(t)$ denote signals with $G(f)$, $G_1(f)$, and $G_2(f)$ denoting their respective Fourier transforms.

(a) ***Superposition theorem*** (linearity):

$$a_1 g_1(t) + a_2 g_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} a_1 G_1(f) + a_2 G_2(f).$$

(b) ***Scale-change*** theorem (scaling property [4, p 88]):

$$g(at) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{|a|} G\left(\frac{f}{a}\right).$$

- The function $g(at)$ represents the function $g(t)$ *compress*ed in time by a factor a (when $|a| > l$). Similarly, the function $G(f/a)$ represents the function $G(f)$ *expand*ed in frequency by the same factor a.
- The scaling property says that if we "squeeze" a function in t, its Fourier transform "stretches out" in f. It is not possible to arbitrarily concentrate both a function and its Fourier transform.
- Generally speaking, the more concentrated $g(t)$ is, the more spread out its Fourier transform $G(f)$ must be.
- This trade-off can be formalized in the form of an *uncertainty principle*. See also 2.28 and 2.29.
- Intuitively, we understand that compression in time by a factor a means that the signal is varying more rapidly by the same factor. To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a. Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed.

(c) ***Duality theorem*** (Symmetry Property [4, p 86]):

$$G(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} g(-f).$$

- In words, for any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of a sign change).

  In particular, if the Fourier transform of $g(t)$ is $G(f)$, then the Fourier transform of $G(f)$ with f replaced by t is the original time-domain signal with t replaced by $-f$.

- If we use the ω-definition (5), we get a similar relationship with an extra factor of 2π:

$$G_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi g(-\omega).$$

Example 2.19. $x(t) = \cos(2\pi a f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(\delta(f - af_0) + \delta(f + af_0)\right).$

Example 2.20. From Example 2.8, we know that

$$1\left[|t| \leq a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2a \ \operatorname{sinc}\left(2\pi a f\right) \tag{15}$$

By the duality theorem, we have

$$2a\operatorname{sinc}(2\pi a t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1[|-f| \leq a],$$

which is the same as

$$\operatorname{sinc}(2\pi f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2f_0}1[|f| \leq f_0]. \tag{16}$$

Both transform pairs are illustrated in Figure 2.

Example 2.21. Let's try to derive the time-shift property from the frequency-shift property. We start with an arbitrary function $g(t)$. Next we will define another function $x(t)$ by setting $X(f)$ to be $g(f)$. Note that f here is just a dummy variable; we can also write $X(t) = g(t)$. Applying the duality theorem to the transform pair $x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X(f)$, we get another transform pair $X(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} x(-f)$. The LHS is $g(t)$; therefore, the RHS must be $G(f)$. This implies $G(f) = x(-f)$. Next, recall the frequency-shift property:

$$e^{j2\pi ct}x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X(f-c).$$

The duality theorem then gives

$$X(t-c) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{j2\pi c-f}x(-f).$$

Replacing $X(t)$ by $g(t)$ and $x(-f)$ by $G(f)$, we finally get the time-shift property.

Definition 2.22. The **convolution** of two signals, $x_1(t)$ and $x_2(t)$, is a new function of time, $x(t)$. We write

$$x = x_1 * x_2.$$

It is defined as the integral of the product of the two functions after one is reversed and shifted:

$$x(t) = (x_1 * x_2)(t) \tag{17}$$

$$= \int_{-\infty}^{+\infty} x_1(\mu)x_2(t-\mu)d\mu = \int_{-\infty}^{+\infty} x_1(t-\mu)x_2(\mu)d\mu. \tag{18}$$

- Note that t is a parameter as far as the integration is concerned.
- The integrand is formed from x_1 and x_2 by three operations:
  (a) time reversal to obtain $x_2(-\mu)$,
  (b) time shifting to obtain $x_2(-(\mu - t)) = x_2(t-\mu)$, and
  (c) multiplication of $x_1(\mu)$ and $x_2(t-\mu)$ to form the integrand.
- In some references, (17) is expressed as $x(t) = x_1(t) * x_2(t)$.

Example 2.23. We can get a triangle from convolution of two rectangular waves. In particular,

$$1[|t| \le a] * 1[|t| \le a] = (2a - |t|) \times 1[|t| \le 2a].$$

2.24. Convolution theorem:

(a) Convolution-in-time rule:

$$x_1 * x_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X_1 \times X_2. \tag{19}$$

(b) Convolution-in-frequency rule:

$$x_1 \times x_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X_1 * X_2. \tag{20}$$

Example 2.25. We can use the convolution theorem to "prove" the frequency-sift property in 2.17.

2.26. From the convolution theorem, we have

- $g^2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G * G$
- if g is band-limited to B, then g^2 is band-limited to $2B$

2.27. Parseval's theorem (Rayleigh's energy theorem, Plancherel formula) for Fourier transform:

$$\int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df. \tag{21}$$

The LHS of (21) is called the (total) **energy** of $g(t)$. On the RHS, $|G(f)|^2$ is called the energy spectral density of $g(t)$. By integrating the energy spectral density over all frequency, we obtain the signal 's total energy. The energy contained in the frequency band B can be found from the integral $\int_B |G(f)|^2 df$.

More generally, Fourier transform preserves the inner product [2, Theorem 2.12]:

$$\langle g_1, g_2 \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) df = \langle G_1, G_2 \rangle.$$

2.28. (Heisenberg) **Uncertainty Principle** [2, 9]: Suppose g is a function which satisfies the normalizing condition $\|g\|_2^2 = \int |g(t)|^2 dt = 1$ which automatically implies that $\|G\|_2^2 = \int |G(f)|^2 df = 1$. Then

$$\left(\int t^2 |g(t)|^2 dt \right) \left(\int f^2 |G(f)|^2 df \right) \geq \frac{1}{16\pi^2}, \tag{22}$$

and equality holds if and only if $g(t) = Ae^{-Bt^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$.

- In fact, we have

  $$\left(\int t^2 |g(t - t_0)|^2 dt \right) \left(\int f^2 |G(f - f_0)|^2 df \right) \geq \frac{1}{16\pi^2},$$

  for every t_0, f_0.

- The proof relies on Cauchy-Schwarz inequality.

- For any function h, define its dispersion Δ_h as $\frac{\int t^2 |h(t)|^2 dt}{\int |h(t)|^2 dt}$. Then, we can apply (22) to the function $g(t) = h(t)/\|h\|_2$ and get

  $$\Delta_h \times \Delta_H \geq \frac{1}{16\pi^2}.$$

2.29. A signal cannot be simultaneously time-limited and band-limited.

Proof. Suppose $g(t)$ is simultaneously (1) time-limited to T_0 and (2) band-limited to B. Pick any positive number T_s and positive integer K such that $f_s = \frac{1}{T_s} > 2B$ and $K > \frac{T_0}{T_s}$. The sampled signal $g_{T_s}(t)$ is given by

$$g_{T_s}(t) = \sum_k g[k]\delta\left(t - kT_s\right) = \sum_{k=-K}^{K} g[k]\delta\left(t - kT_s\right)$$

where $g[k] = g\left(kT_s\right)$. Now, because we sample the signal faster than the Nyquist rate, we can reconstruct the signal g by producing $g_{T_s} * h_r$ where the LPF h_r is given by

$$H_r(\omega) = T_s 1[\omega < 2\pi f_c]$$

with the restriction that $B < f_c < \frac{1}{T_s} - B$. In frequency domain, we have

$$G(\omega) = \sum_{k=-K}^{K} g[k]e^{-jk\omega T_s} H_r(\omega).$$

Consider ω inside the interval $I = (2\pi B, 2\pi f_c)$. Then,

$$0 \overset{\omega > 2\pi B}{=} G(\omega) \overset{\omega < 2\pi f_c}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) e^{-jk\omega T_s} \overset{z = e^{j\omega T_s}}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) z^{-k} \tag{23}$$

Because $z \neq 0$, we can divide (23) by z^{-K} and then the last term becomes a polynomial of the form

$$a_{2K} z^{2K} + a_{2K-1} z^{2K-1} + \cdots + a_1 z + a_0.$$

By fundamental theorem of algebra, this polynomial has only finitely many roots– that is there are only finitely many values of $z = e^{j\omega T_s}$ which satisfies (23). Because there are uncountably many values of ω in the interval I and hence uncountably many values of $z = e^{j\omega T_s}$ which satisfy (23), we have a contradiction. $\square$